EPFL– Fall 2025
SOLUTIONS: Series 7

Differential Geometry IV: General Relativity

G. Moschidis
5 Nov. 2025

7.1 For the following spacetimes, decide if they are globally hyperbolic or not. If they are, find a Cauchy hypersurface.

(a) The future timecone

$$I_+[0] = \big\{(x^0,\dots,x^n):\ -(x^0)^2+(x^1)^2+\dots+(x^n)^2<0 \quad \text{and} \quad x^0>0\big\}$$

inside Minkowski spacetime (equipped with the Minkowski metric η).

(b) The spacetime $(\mathbb{R}^{1+3})$ equipped with a Lorentzian metric g which satisfies, in the Cartesian coordinates,

$$|g_{\alpha\beta}-\eta_{\alpha\beta}|<\frac{1}{10}$$

(c) The $1+1$ dimensional Anti-de Sitter spacetime from Exercise 6.3.

Solution. (a) We will show that $(I_+[0],\eta)$ is globally hyperbolic, and the usual hyperboloidal foliation provides a time function with level sets which are Cauchy hypersurfaces. To this end, let us define the function $\tau : I_+[0]\to(0,+\infty)$ by

$$\tau \doteq (x^0)^2-\sum_{i=1}^n (x^i)^2.$$

Note that $(I_+[0],\eta)$ is trivially time oriented; we can choose the time orientation for which ∂_0 is future directed. Note also that the hypersurfaces $\{\tau=\text{const}$ are spacelike; see the proof of Exercise 2.1.

If $\gamma(s)=(x^0(s),\dots,x^n(s))$ is any future directed causal curve, then $\tau(\gamma(s))$ is strictly increasing in s, since

$$\frac{d}{ds}\tau(\gamma(s))=\partial_\alpha\tau|_{\gamma(s)}\cdot\dot{x}^\alpha(s)=2x^0(s)\dot{x}^0(s)-2\sum_{i=1}^n x^i(s)\dot{x}^i(s)=2\eta_{\alpha\beta}x^\alpha(s)\dot{x}^\beta(s)=2\eta\big(\gamma(s),\dot\gamma(s)\big)<0$$

the last inequality following from the fact that $\gamma(s)\in I_+[0]$ and γ was assumed to be causal and future directed, so $\dot\gamma(s)\in J_+[0]$. Therefore, for any $\tau_0\in(0,+\infty)$, each future directed causal curve can intersect $\{\tau=\tau_0\}$ at most once.

It remains to show that, for each $\tau_0\in(0,+\infty)$, each inextendible causal curve $\gamma:(a,b)\to I_+[0]$ (with $a,b\in(-\infty,+\infty)$; note that we can alsways reparametrize a curve so as to have a domain of finite range) intersects the level set $\{\tau=\tau_0\}$. Without loss of generality (by switching the parametrization, if necessary) we can assume that γ is future directed. Let $s_1\in(a,b)$ and set

$$\tau_1=\tau(\gamma(s_1)).$$

Again, without loss of generality, we can assume that $\tau_1<\tau_0$ (otherwise, we apply the same arguments as below, just working towards the past).

Let us assume, for the sake of contradiction, that γ does not intersect $\{\tau = \tau_0\}$; since $\gamma(s_1) \in \{\tau < \tau_0\}$, this means that γ is contained in the set $\{\tau < \tau_0\}$. Moreover, since γ is causal and future directed, we have that $\gamma(s) \in J^+[\gamma(s_1)]$ for all $s \geqslant s_1$. Therefore, $\gamma|_{[s_1,b)}$ is contained inside the set

$$\mathcal{K} = J^+[\gamma(s_1)] \cap \{\tau \leqslant \tau_0\}$$

which is a *compact* subset of $I^+[0]$. As we will show below, a causal curve that remains in a compact subset has to be extendible, which is a contradiction.

Proof of extendibility of causal curves staying in a compact set: Let $(\mathcal{M}^{n+1}, g)$ be a spacetime and $\mathcal{K} \subset \mathcal{M}$ be a *compact* subset. Let $\gamma : (0,1) \to \mathcal{M}$, be a continuous future directed causal curve that is contained inside $\mathcal{K}$. We will show that there exists a continuous future directed causal curve $\tilde{\gamma} : (0, 1+\epsilon) \to \mathcal{M}$ extending γ, i.e. $\tilde{\gamma}(s) = \gamma(s)$ for all $s \in (0,1)$.

Since $\mathcal{K}$ is compact, there exists a sequence $s_k \in (0,1)$, $s_k \xrightarrow{k\to\infty} 1$, and a point $q \in \mathcal{K}$, such that the sequence $p_k = \gamma(s_k)$ converges to q. For $\epsilon_0 > 0$ sufficiently small, there exists a small neighborhood $\mathcal{U}$ of q with a local coordinate system $(x^0, \dots, x^n)$ in which the metric g satisfies

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| < \epsilon_0 \tag{1}$$

and q has coordinates $(0, \dots, 0)$ (normal coordinates around q, for instance, have these properties on a coordinate ball of sufficiently small radius).

In view of (1), the fact that γ is causal implies that, if $\gamma(z), \gamma(w) \in \mathcal{U}$, then their coordinates satisfy

$$\frac{|x^i(\gamma(z)) - x^i(\gamma(w))|}{|x^0(\gamma(z)) - x^0(\gamma(w))|} \leqslant 1 + O(\epsilon_0) \leqslant 2 \quad \text{for all} \quad i = 1, \dots, n.$$

In particular, this means that the curve $\gamma \cap \mathcal{U}$ can be parametrized by the x^0 coordinate and, with that parametrization, the corresponding curve $t \to \gamma(t) = \big(t, x^1(t), \dots, x^n(t)\big)$ is a *Lipschitz* curve in $\mathbb{R}^{n+1}$ (with respect to the coordinate distance).

If we set $t_k = x^0(p_k)$ (for k sufficiently large so that p_k is inside $\mathcal{U}$), then our assumption that $p_k \to q$ translates to $\gamma(t_k) = (t_k, x^1(t_k), \dots, x^n(t_k)) \to (0, \dots, 0)$. Since $\gamma(t)$ is Lipschitz, this means that $\gamma(t) \to (0, \to 0)$ as $t \to 0$ (i.e. convergence of a sequence implies that the whole curve converges). Therefore, if $\sigma : [0,1) \to U$ is any future directed causal curve with $\sigma(0) = q$, then the combined curve

$$\tilde{g}(t) = \begin{cases} \gamma(t), & t < 0, \\ \sigma(t), & t \geqslant 0 \end{cases}$$

is continuous, causal and future directed.

(b) The spacetime $(\mathbb{R}^{3+1}, g)$ is also globally hyperbolic and the level sets $\{x^0 = const\}$ are all Cauchy hypersurfaces. In order to see this, note that the condition $|g_{\alpha\beta} - \eta_{\alpha\beta}| \leqslant \frac{1}{10}$ implies that, if γ is a causal curve, then

$$0 \geqslant g_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta = \eta_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta + (g_{\alpha\beta} - \eta_{\alpha\beta})\dot{\gamma}^\alpha\dot{\gamma}^\beta = -(\dot{\gamma}^0)^2 + \sum_{i=1}^{3}(\dot{\gamma}^i)^2 + (g_{\alpha\beta} - \eta_{\alpha\beta})\dot{\gamma}^\alpha\dot{\gamma}^\beta$$

$$\geqslant -2(\dot{\gamma}^0)^2 + \frac{1}{2}\sum_{i=1}^{3}(\dot{\gamma}^i)^2,$$

i.e.

$$|\dot{\gamma}^0| \geqslant \frac{1}{2}\sqrt{\sum_{i=1}^{3}(\dot{\gamma}^i)^2}. \tag{2}$$

In particular, the x^0 coordinate is strictly monotonic along γ and, hence, x^0 can be used to parametrize the curve and, moreover, γ intersects $\{x^0 = const\}$ at most once.

We will now show that an inextendible causal curve $\gamma : (a,b) \to \mathbb{R}^{3+1}$ intersects every $\{x^0 = const\}$ hypersurface at least once; without loss of generality, we will consider the case of $\{x^0 = 0\}$ and, as in part (a), we will assume that γ is future directed and that $\gamma(s_1) \in \{x^0 < 0\}$ for some $s_1 \in (a,b)$. If we assume for the sake of contradiction that $\gamma \cap \{x^0 = 0\} = \emptyset$, then

$$\gamma|_{[s_1,b)} \subset J^+[\gamma(s_1)] \cap \{x^0 \leqslant 0\}.$$

However, in view of the fact that any causal curve emanating from $\gamma(s_1)$ satisfies (2), it follows that

$$J^+[(p^0,p^1,p^2,p^3)] \subseteq \Big\{(x^0,x^1,x^2,x^3) : x^0 - p^0 \geqslant \frac{1}{2}\sqrt{\sum_{i=1}^{3}(x^i-p^i)^2} \quad \text{and} \quad x^0 \geqslant p^0\Big\},$$

i.e. $J^+[(p^0,p^1,p^2,p^3)]$ is contained in the future of cone of “twice” the width of that associated to η. Thus, $J^+[\gamma(s_1)] \cap \{x^0 \leqslant 0\}$ is contained in a compact set. Using the lemma proved at the end of part (a), this implies that γ cannot be inextendible, which is a contradiction.

(c) The AdS spacetime is not globally hyperbolic. As we proved in Exercise 6.b, there exist points p, q in that spacetime with $q \in I^+[p]$ such that no timelike geodesic exists connecting p, q; recall that, on a globally hyperbolic spacetime, such a timelike geodesic always exist. You can also check that, with the notations of Exercise 6,b, the causal diamond $J^+[(0,0)] \cap J^-[(\pi,0)]$ is not compact (extends all the way to “infinity” in the x direction).

7.2 Let $(\mathcal{M}, g)$ be a spacetime and let $p \in \mathcal{M}$.

(a) Show that if $q \in J^+(p)$, then there exists a sequence of points $q_n \in I^+(p)$ with $q_n \xrightarrow{n\to\infty} q$, i.e.

$$J^+(p) \subset \text{clos}\big(I^+(p)\big).$$

Hint: Starting from a causal curve γ connecting p to q, you need to find a sequence of timelike curves γ_n emanating from p converging to γ. To this end, if T is a globally timelike vector field on $\mathcal{M}$, consider variations $\gamma_s(t)$ of $\gamma(t) = \gamma_0(t)$ such that the variation vector field $\frac{\partial}{\partial s}(\gamma_s(t))|_{s=0}$ is of the form $f(t)T|_{\gamma(t)}$ for an appropriately chosen function f.

(b) Assume, moreover, that $(\mathcal{M}, g)$ is *globally hyperbolic*. Prove that, in this case

$$J^+(p) = \text{clos}\big(I^+(p)\big).$$

(c) Can you find an example of a (necessarily not globally hyperbolic) spacetime $(\mathcal{M}, g)$ with a point $p \in \mathcal{M}$ such that $J^+(p)$ is not closed?

Solution. (a) Let $\gamma : [0,1] \to \mathcal{M}$ be a future directed causal curve such that $\gamma(0) = p$ and $\gamma(1) = q$. Let N be a a future directed timelike vector field on $\mathcal{M}$ and $f : [0,1] \to [0,+\infty)$ a function that we will determine shortly, satisfying the condition

$$f(0) = 0.$$

Let also $\gamma_s : [0,1]$, $s \in [0,1)$ be a family of curves which is a variation of γ (i.e. satisfy $\gamma_0 = \gamma$) with

$$\gamma_s(0) = \gamma(0) = p$$

and with variation vector field X which satisfies

$$X|_\gamma = f \cdot N$$

(since $f(0) = 0$, the above two requirements are consistent). We will show that, for $s > 0$ small enough, the tangent $\dot{\gamma}_s$ is future directed *timelike*; this will imply that the points $\gamma_s(1)$ belong to $I^+[p]$; in view of the fact that $\gamma_s(1) \xrightarrow{s\to 0} \gamma_0(1) = q$, this will imply that $q \in clos(I^+[p])$, as required.

For any $t_0 \in [0,1]$, let $(x^0, \dots, x^n)$ be a local coordinate chart around $\gamma(t_0)$ such that ∂_0 is future directed and timelike; then

$$N^0|_{\gamma(t)} > 0.$$

With respect to those coordinates, we can calculate for any $t \in [0,1]$ such that $\gamma(t)$ lies inside this coordinate chart:

$$\frac{\partial}{\partial s}\gamma_s^\alpha(t)\big|_{s=0} = X^i|_{\gamma(t)} = f(t)N^i|_{\gamma(t)}$$

and, therefore, by differentiating in t:

$$\frac{\partial}{\partial s}\dot{\gamma}_s^\alpha(t)\big|_{s=0} = f'(t)N^i|_{\gamma(t)} + f(t)\frac{dN^i|_{\gamma(t)}}{dt}.$$

Therefore, integrating in s, we have for s small enough:

$$\dot{\gamma}_s^\alpha(t) = \dot{\gamma}^\alpha(t) + s\big(f'(t)N^i|_{\gamma(t)} + f(t)\frac{dN^i|_{\gamma(t)}}{dt}\big) + O(s^2).$$

Since $\dot{\gamma}$ is future directed and causal, the above vector will be future directed and timelike as long as $f'(t)N^i|_{\gamma(t)} + f(t)\frac{dN^i|_{\gamma(t)}}{dt}$ is future directed and timelike and s is small enough. For this, it suffices to fix f to satisfy

$$f'(t) \geqslant Cf(t) \quad \text{for all } t \in [0,1]$$

for a constant C which is large enough in terms of the ratios $\left|\frac{\frac{dN^i|_{\gamma(t)}}{dt}}{N^0|_{\gamma(t)}}\right|$ and $\left|\frac{N^i|_{\gamma(t)}}{N^0|_{\gamma(t)}}\right|$. In each coordinate chart as above, the fact that such a constant exists follows by the fact that N is continuously differentiable along γ; since $\gamma([0,1])$ is a compact set in $\mathcal{M}$, it can be covered by a finite number of coordinate charts as above, so this constant C can be chosen to be the same for each of these coordinate charts.

(b) Let $q \in clos\big(I^+[p]\big)$; by the construction of part (a), there exist a sequence $q_n \in I^+[p]$ such that $q_n \to q$ and which, moreover, satisfies the property that $q_{n+1} \in I^-[q_n]$; this is because the curve $s \to \zeta(s) = \gamma_s(1)$ constructed in part (a) is timelike future directed for small s, since $\partial_s \gamma_s(1)|_{s=0} = f(1)N|_{\gamma(1)}$. In particular, if σ is any future directed causal curve connecting p to q_n, I can extend it to a future directed causal curve $\tilde{\sigma}$ connecting p to q_0 by following the curve $\zeta(s)$ from q_n to q_0. In particular, denoting with C_n the space of curves

$$C_n \doteq \Big\{\sigma : [0,1] \to \mathcal{M},\, \sigma(0) = p,\, \sigma(1) = q_n,\, \gamma \text{ is continuous, causal and future directed}\Big\},$$

by extending each curve σ in C_n connecting p to q_n to a curve $\tilde{\sigma}$ as described above connecting p to q_0, we can homeomorphically identify C_n with the subset $\tilde{C}_n$ of C_0 of future directed causal curves going from p to q_n and from there to q by following the curve ζ. In particular,

$$\tilde{C}_{n+1} \subseteq \tilde{C}_n.$$

In particular, each curve in $\tilde{C}_n$ is a future directed causal curve connecting p to q_0 passing through the points $q_1, \dots, q_n$

Since $(\mathcal{M}, g)$ is globally hyperbolic, the space of curves C_n is *compact* with the C^0 topology. In particular, $\tilde{C}_n$ is a compact subset of C_0. Therefore, the monotonicity property above implies that

$$\tilde{C} \doteq \bigcap_{n=0}^{\infty} \tilde{C}_n$$

is non-empty and compact. But each curve σ in $\tilde{C}$ is a future directed causal curve connecting p to q_0 such that σ passes through all the points q_n, $n \in \mathbb{N}$. By continuity, this means that σ also passes through $q = \lim_n q_n$. Hence, q is connected to p through a future directed causal curve and, thus, $q \in J^+[p]$. Thus,

$$clos\big(I^+[p]\big) \subseteq J^+[p].$$

Combined with part (a), we obtain the equality of the two sets.

(c) Consider $(\mathbb{R}^{1+1}, \eta)$ with the point $(1,1)$ removed. Then $J^+[(0,0)]$ consists of the cone $\{(t,x) : t \geqslant |x|\}$ with the ray $\{(t,t) : \quad t \geqslant 1\}$ removed.

7.3 In this exercise, we will explore some of the geometric properties of the Riemann curvature tensor. To this end, let us fix a smooth Lorentzian manifold $(\mathcal{M}, g)$. Recall that

$$R(X,Y)Z \doteq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

(A) Let $\phi : (-\epsilon, \epsilon) \times [0,1] \to \mathcal{M}$ be a smooth map such that, for each $s \in (-\epsilon, \epsilon)$, $\gamma_s = \phi(s, \cdot)$ is a *geodesic*. Define the vector fields $T = d\phi(\frac{\partial}{\partial t})$ and $X = d\phi(\frac{\partial}{\partial s})$.

(a) Prove that $[T, X] = 0$. *(Hint: Compare $\nabla_X T$ and $\nabla_T X$.)*

(b) Let us define the acceleration vector field

$$a = \nabla_T \nabla_T X.$$

Prove that

$$a = -R(X,T)T.$$

Intuitively, X measures the infinitesimal separation between nearby geodesics; thus, when the right hand side above is non-zero, nearby geodesics tend to accelerate towards or away from each other.

(B) Let $\gamma : [0,1] \to \mathcal{M}$ be a smooth curve. For any $t_1, t_2 \in [0,1]$, we will denote with $\mathbb{P}_{\gamma(t_1)\to\gamma(t_2)} : T_{\gamma(t_1)}\mathcal{M} \to T_{\gamma(t_2)}\mathcal{M}$ the parallel transport along γ from $\gamma(t_1)$ to $\gamma(t_2)$.

(a) Prove that, for any vector field Z along γ, as $\tau \to 0$:

$$\lim_{\tau\to 0} \frac{Z|_{t=0} - \mathbb{P}_{\gamma(\tau)\to\gamma(0)} Z|_{t=\tau}}{\tau} = -\nabla_{\dot\gamma(0)} Z.$$

Hint: Contruct a frame $\{e_i\}_{i=1}^n$ *of vector fields along* γ *which are parallel translated, and express* Z *in components with respect to* e_i*.*

*(b) Let $\phi : [-1,1]\times[-1,1] \to \mathcal{M}$ be a smooth map with $p = \phi(0,0)$ and let $X = \phi^*(\frac{\partial}{\partial x^1})$ and $Y = \phi^*(\frac{\partial}{\partial x^2})$. For any $s_1, s_2 \in (0,1)$, we will consider the rectangular loop $\gamma_{(s_1,s_2)}$ starting and ending at p which is of the form $\gamma_{(s_1,s_2)} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned} \gamma_1(t) &= \phi(t,0), \quad t \in [0,s_1], \\ \gamma_2(s) &= \phi(s_1,s), \quad s \in [0,s_2], \\ \gamma_3(t) &= \phi(s_1 - t, s_2), \quad t \in [0,s_1], \\ \gamma_3(s) &= \phi(0, s_2 - s), \quad s \in [0,s_2]. \end{aligned}$$

For any $Z \in T_p\mathcal{M}$, let $Z_{(s_1,s_2)} \in T_p\mathcal{M}$ be the tangent vector obtained after parallel transporting Z_p around γ, i.e. following the successive mappings

$$\begin{aligned} Z \to & Z' = \mathbb{P}_{\gamma_1(0)\to\gamma_1(s_1)} Z \to Z'' = \mathbb{P}_{\gamma_2(0)\to\gamma_2(s_2)} Z' \\ & \to Z''' = \mathbb{P}_{\gamma_3(0)\to\gamma_3(s_1)} Z'' \to Z_{(s_1,s_2)} = \mathbb{P}_{\gamma_4(0)\to\gamma_4(s_2)} Z'''. \end{aligned}$$

Show that

$$\lim_{s_2\to 0}\lim_{s_1\to 0} \frac{Z_{(s_1,s_2)} - Z}{s_1 s_2} = -R(X,Y)Z.$$

Solution. (A) (a) In any local coordinate system $(x^0,\ldots,x^n)$ on $\mathcal{M}$, we have

$$T^\alpha = \frac{\partial\phi^\alpha}{\partial t} \quad \text{and} \quad X^\alpha = \frac{\partial\phi^\alpha}{\partial s}$$

Note that, for any $s \in (-\epsilon,\epsilon)$, we can view X as a vector field along the curve $\gamma_s : [0,1] \to \mathcal{M}$, $\gamma_s(t) = \phi(s,t)$, whose tangent vector is T; thus, we can compute

$$(\nabla_T X)^\alpha = \frac{d}{dt}\big(X^\alpha|_{\gamma_s(t)}\big) + \Gamma^\alpha_{\beta\gamma}|_{\gamma_s(t)} T^\beta X^\gamma = \frac{\partial^2\phi^\alpha}{\partial t\partial s} + \Gamma^\alpha_{\beta\gamma}|_{\phi(s,t)} \frac{\partial\phi^\beta}{\partial t}\frac{\partial\phi^\gamma}{\partial s}.$$

Similarly, for any $t \in [0,1]$, we can think of T as a vector field along the curve $\phi_t : (-\epsilon, \epsilon) \to \mathcal{M}$, $\phi_t(s) = \phi(s,t)$, whose tangent vector is X; we therefore have:

$$(\nabla_X T)^\alpha = \frac{d}{ds}\big(T^\alpha|_{\phi_t(s)}\big) + \Gamma^\alpha_{\beta\gamma}|_{\phi_t(s)} X^\beta T^\gamma = \frac{\partial^2 \phi^\alpha}{\partial t \partial s} + \Gamma^\alpha_{\beta\gamma}|_{\phi(s,t)} \frac{\partial \phi^\beta}{\partial s}\frac{\partial \phi^\gamma}{\partial t}.$$

Using the fact that the Levi-Civita connection is torsion-free and $\Gamma^\alpha_{\beta\gamma}$ is symmetric in β, γ, we can readily compute

$$[T,X]^\alpha = (\nabla_T X)^\alpha - (\nabla_X T)^\alpha = 0.$$

(b) Using the fact that $[X,T] = \nabla_X T - \nabla_T X = 0$ (from part (a)) and the fact that $R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$, we can readily calculate:

$$a = \nabla_T \nabla_T X = \nabla_T \nabla_X T = -R(T,X)T + \nabla_X \nabla_T T - \nabla_{[T,X]} T = -R(T,X)T,$$

where, in passing to the last equality above, we made use of the fact that γ_s is a geodesic and $T = \dot{\gamma}_s$, so that $\nabla_T T = 0$.

(B) (a) Let $\{\xi_\alpha\}_{\alpha=0}^n$ be a basis of orthonormal tangent vectors in $T_{\gamma(0)}\mathcal{M}$ with respect to $g|_{\gamma(0)}$ and let $\{e_\alpha\}_{\alpha=0}^n$ be a set of vector fields along γ such that e_α is the parallel translate of ξ_α (i.e. $e_\alpha|_{t=0} = e_\alpha$ and $\nabla_{\dot\gamma} e_\alpha = 0$. Since

$$\frac{d}{dt} g(e_\alpha, e_\beta)|_{\gamma(t)} = g(\nabla_{\dot\gamma} e_\alpha, e_\beta) + g(e_\alpha, \nabla_{\dot\gamma} e_\beta) = 0,$$

we infer that, for any $t \in [0,1]$, $\{e_\alpha|_{\gamma(t)}\}_{\alpha=0}^n$ is an orthonormal base for $T_{\gamma(t)}\mathcal{M}$.

Any vector field Z along γ can be expressed, with respect to the basis $\{e_\alpha\}_{\alpha=0}^n$ as $Z = Z^\alpha e_\alpha$ for some (unique) component functions $Z_\alpha : [0,1] \to \mathbb{R}$, $\alpha = 0, \dots, n$. In this basis, the covariant derivative and the parallel translation of a vector field become a standard derivative and translation, respectively, of the component functions; in particular, we can readily compute:

$$\nabla_{\dot\gamma} Z = \nabla_{\dot\gamma}(Z^\alpha e_\alpha) = \frac{dZ^\alpha}{dt} e_\alpha + Z^\alpha \nabla_{\dot\gamma} e_\alpha = \frac{dZ^\alpha}{dt} e_\alpha.$$

Moreover, since, for any $t_1, t_2 \in [0,1]$, we have $\mathbb{P}_{\gamma(t_1)\to\gamma(t_2)} e_\alpha|_{\gamma(t_1)} = e_\alpha|_{\gamma(t_2)}$, the linearity of the parallel transport operator implies that if $v = v^\alpha e_\alpha|_{\gamma(t_1)}$ is an element of $T_{\gamma(t_1)}\mathcal{M}$, then

$$\mathbb{P}_{\gamma(t_1)\to\gamma(t_2)} v = v^\alpha e_\alpha|_{\gamma(t_2)}.$$

We can thus calculate:

$$\begin{aligned}
\lim_{\tau\to 0} \frac{Z|_{t=0} - \mathbb{P}_{\gamma(\tau)\to\gamma(0)} Z|_{t=\tau}}{\tau} &= \lim_{\tau\to 0} \frac{Z^\alpha(0) e_\alpha|_{t=0} - \mathbb{P}_{\gamma(\tau)\to\gamma(0)}(Z^\alpha(\tau) e_\alpha|_{t=\tau})}{\tau} \\
&= \lim_{\tau\to 0} \frac{Z^\alpha(0) e_\alpha|_{t=0} - Z^\alpha(\tau) e_\alpha|_{t=0}}{\tau} \\
&= -\frac{dZ^\alpha}{dt}(0) e_\alpha \\
&= -\nabla_{\dot\gamma(0)} Z.
\end{aligned}$$

Moreover, using Taylor's theorem to express for any $t \in [0,1]$:

$$Z^\alpha(t) = Z^\alpha(0) + \frac{dZ^\alpha}{dt}(0)t + \frac{1}{2}\frac{d^2 Z^\alpha}{dt^2}(\xi(t))t^2$$

for some $\xi(t) \in [0,t]$ depending smoothly on t, we also have the following useful expression for the parallel transport operator:

$$\mathbb{P}_{\gamma(t)\to\gamma(0)} Z = Z|_{\gamma(0)} + \nabla_{\dot\gamma(0)} Z \cdot t + V[t] \cdot t^2 \tag{3}$$

for some smooth function $V : t \to V[t] \in T_{\gamma(0)}\mathcal{M}$ with $V^\alpha[t] = \frac{1}{2}\frac{d^2 Z^\alpha}{dt^2}(\xi(t))$; note that

$$V[t] \xrightarrow{t\to 0} \frac{1}{2}\nabla_{\dot\gamma}\nabla_{\dot\gamma} Z|_{t=0}.$$

(b) For $\phi : [-1,1]\times[-1,1] \to \mathcal{M}$ as in the statement of the exercise, we will denote by $\gamma_s(\cdot)$ the family of curves $t \to \phi(s,t)$ and by $\gamma'_t(\cdot)$ the family of curves $s \to \phi(s,t)$ in $\mathcal{M}$. Note that $X|_{\phi(s,t)} = \dot\gamma_s(t)$ and $Y|_{\phi(s,t)} = \dot\gamma'_t(s)$. For any vector field W defined along the image of the map ϕ and any $h \in (-1,1)$, we will define the vector fields $\mathbb{P}^{(h)}W$ on $\phi\big([-1,1]\times[-1+|h|, 1-|h|]\big)$ and $\mathbb{P}'^{(h)}W$ on $\phi\big([-1+|h|, 1-|h|]\times[-1,1]\big)$ to be the parallel translates of W along γ_s and γ'_t, respectively, with step h, i.e.

$$\big(\mathbb{P}^{(h)}W\big)|_{\phi(s,t)} = \mathbb{P}_{\gamma_s(t-h)\to\gamma_s(t)} W \quad \text{and} \quad \big(\mathbb{P}'^{(h)}W\big)|_{\phi(s,t)} = \mathbb{P}_{\gamma'_t(s-h)\to\gamma'_t(s)} W.$$

Note that, applying (3) for γ_s and γ'_t, we obtain

$$\begin{aligned}\big(\mathbb{P}^{(h)}W\big)|_{\phi(s,t)} &= W|_{\phi(s,t)} - (\nabla_X W)|_{\phi(s,t)} h + V_1[W;h]|_{\phi(s,t)} h^2 \\ \big(\mathbb{P}'^{(h)}W\big)|_{\phi(s,t)} &= W|_{\phi(s,t)} - (\nabla_Y W)|_{\phi(s,t)} h + V_2[W;h]|_{\phi(s,t)} h^2\end{aligned}$$

for some smooth vector fields $V_1[W;h], V_2[W;h]$ depending smoothly on h and W and satisfying

$$V_1[W,h] \xrightarrow{h\to 0} \frac{1}{2}\nabla_X\nabla_X W, \quad V_2[W,h] \xrightarrow{h\to 0} \frac{1}{2}\nabla_Y\nabla_Y W. \tag{4}$$

Using the above formulas, we can compute for any $s_1, s_2 > 0$

$$\begin{aligned}\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z &= \mathbb{P}'^{(s_2)}\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big) \\ &= \big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big) - \Big(\nabla_Y\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big)\Big)s_2 \\ &\qquad\qquad + V_2\big[\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big); s_2\big]s_2^2 \\ &= Z - (\nabla_X Z)s_1 - (\nabla_Y Z)s_2 + (\nabla_Y\nabla_X Z)s_1 s_2 \\ &\qquad + V_1[Z;s_1]s_1^2 + V_2\big[\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big); s_2\big]s_2^2 - (\nabla_Y V_1[Z;s_1])s_1^2 s_2\end{aligned}$$

and, similarly,

$$\mathbb{P}^{(s_1)}\mathbb{P}'^{(s_2)}Z = Z - (\nabla_X Z)s_1 - (\nabla_Y Z)s_2 + (\nabla_X\nabla_Y Z)s_1 s_2$$

$$+ V_2[Z; s_2]s_2^2 + V_1\big[\big(Z - (\nabla_Y Z)s_2 + V_2[Z; s_2]s_2^2\big); s_2\big]s_1^2 - (\nabla_X V_2[Z; s_2])s_2^2 s_1.$$

Therefore, we compute

$$\begin{aligned}
&\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z - \mathbb{P}^{(s_1)}\mathbb{P}'^{(s_2)}Z \\
&\quad = \big(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z\big)s_1 s_2 \\
&\qquad + \Big(V_1[Z; s_1] - V_1\big[\big(Z - (\nabla_Y Z)s_2 + V_2[Z; s_2]s_2^2\big); s_2\big]\Big)s_1^2 \\
&\qquad + \Big(V_2\big[\big(Z - (\nabla_X Z)s_1 + V_1[Z; s_1]s_1^2\big); s_2\big] - V_2[Z; s_2]\Big)s_2^2 \\
&\qquad - \Big((\nabla_Y V_1[Z; s_1]) - (\nabla_X V_2[Z; s_2])\Big)s_2^2 s_1.
\end{aligned}$$

In particular, using (4) for the second and third lines in the right hand side, we have:

$$\lim_{(s_1,s_2)\to(0,0)} \frac{\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z - \mathbb{P}^{(s_1)}\mathbb{P}'^{(s_2)}Z}{s_1 s_2} = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z = R(Y, X)Z.$$

Using the fact that $\mathbb{P}^{(-h)}\mathbb{P}^{(h)} = \mathrm{Id}$ (and similarly for $\mathbb{P}'$), we have

$$\begin{aligned}
Z_{(s_1,s_2)}|_p - Z|_p &= \big(\mathbb{P}'^{(-s_2)}\mathbb{P}^{(-s_1)}\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z\big)|_{\phi(0,0)} - Z|_{\phi(0,0)} \\
&= \Big(\mathbb{P}'^{(-s_2)}\mathbb{P}^{(-s_1)}\Big[\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z - \mathbb{P}^{(s_1)}\mathbb{P}^{('s_2)}Z\Big]\Big)\Big|_{\phi(0,0)}
\end{aligned}$$

Therefore, since $\lim_{h\to 0}\mathbb{P}^{(-h)} = \mathrm{Id}$ (and similarly for $\mathbb{P}'$), we obtain the required formula:

$$\lim_{(s_1,s_2)\to(0,0)} \frac{Z_{(s_1,s_2)}|_p - Z|_p}{s_1 s_2} = R(Y, X)Z|_p.$$

7.4 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold of dimension $n+1$ and let R be its Riemann curvature tensor.

(a) Show that, in any local coordinate chart $(x^0, \dots, x^n)$ on $\mathcal{M}$, the components of R satisfy the following identities:

1. $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$.
2. $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$.
3. $R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$ (*First Bianchi identity*).
4. $\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} = 0$ (*Second Bianchi identity*).

Prove that the Ricci tensor satisfies:

$$g^{\alpha\beta}\nabla_\alpha\big(Ric_{\beta\gamma} - \frac{1}{2}Rg_{\beta\gamma}\big) = 0.$$

That is to say, the Einstein tensor of every Lorentzian manifold is *divergence free.*

(b) Prove that, when $n+1=3$, $R_{\alpha\beta\gamma\delta}$ has exactly 6 independent components. Noting that this is the same number of independent components as for the Ricci tensor $Ric_{ij} = g^{ab}R_{iajb}$, can you prove that, when $n+1=3$, $Ric_{ij}=0$ implies that $R_{ijkl}=0$? How many independent components do these tensors have in dimension $n+1=2$?

(c) When $n+1 \geqslant 3$, define the Weyl tensor by the relation

$$\begin{aligned} W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} &+ \frac{1}{n-1}\Big(Ric_{\alpha\delta}g_{\beta\gamma} - Ric_{\alpha\gamma}g_{\beta\delta} + Ric_{\beta\gamma}g_{\alpha\delta} - Ric_{\beta\delta}g_{\alpha\gamma}\Big) \\ &+ \frac{1}{n(n-1)}R\big(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\big) \end{aligned}$$

where $R = g^{ij}Ric_{ij}$ is the Ricci scalar. Prove that the Weyl tensor satisfies the same symmetries as the Riemann tensor, and moreover

$$g^{\alpha\gamma}W_{\alpha\beta\gamma\delta} = 0.$$

Deduce that $W_{\alpha\beta\gamma\delta}=0$ when $n+1=3$.

*(d) Let $\phi : \mathcal{M} \to \mathbb{R}_+$ be a C^∞ function and consider the conformal metric

$$\tilde{g} = \phi^2 g.$$

Show that the Weyl tensors of g and $\tilde{g}$ satisfy

$$\tilde{W}^{\alpha}{}_{\beta\gamma\delta} = W^{\alpha}{}_{\beta\gamma\delta},$$

i.e. W is a *conformal invariant* of g. Deduce that a necessary condition for a metric g to be conformally flat, i.e. of the form $\phi^2\eta$, is that $W=0$ (it can be shown that it is also a sufficient condition when $\dim\mathcal{M} > 3$).

Solution. (a) All of the above identities are tensorial in nature, i.e. can be reexpressed in a coordinate independent way as follows: For any vector fields X, Y, Z, W on $\mathcal{M}$,

$$\begin{aligned} R(X,Y)Z &= -R(Y,X)Z, \\ g\big(R(X,Y)Z,W\big) &= -g\big(R(Z,W)X,Y\big), \\ R(X,Y)Z + R(Z,X)Y + R(Y,Z)X &= 0, \\ (\nabla_Z R)(X,Y)W + (\nabla_Y R)(Z,X)W + (\nabla_X R)(Y,Z)W &= 0. \end{aligned}$$

Thus, in order to prove these identities, it suffices to show that they are true in one coordinate system.

The most convenient coordinates to establish pointwise identities are the normal ones: Recall that, for any $p \in \mathcal{M}$, if $(x^0,\dots,x^n)$ are normal coordinates around p, then

$$g_{\alpha\beta}|_p = \eta_{\alpha\beta} \quad \text{and} \quad \partial_\alpha g_{\beta\gamma}|_p = 0 = \Gamma^\lambda_{\alpha\beta}|_p. \tag{5}$$

Moreover, as a consequence of the lemma of Gauss:

$$\partial^2_{\alpha\beta} g_{\gamma\delta}|_p + \partial^2_{\gamma\alpha} g_{\beta\delta}|_p + \partial^2_{\beta\gamma} g_{\alpha\delta}|_p = 0. \tag{6}$$

Therefore, for any tensor field T around p:

$$(\nabla_\alpha T)^{i_1\dots i_k}_{j_1\dots j_l}|_p = \partial_\alpha T^{i_1\dots i_k}_{j_1\dots j_l}|_p.$$

Moreover, recall that, in any coordinate system, the Riemann curvature tensor takes the form

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda}\big(\partial_\gamma \Gamma^\lambda_{\delta\beta} - \partial_\delta \Gamma^\lambda_{\gamma\beta} + \Gamma^\lambda_{\gamma\rho}\Gamma^\rho_{\delta\beta} - \Gamma^\lambda_{\delta\rho}\Gamma^\rho_{\gamma\beta}\big),$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\big(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\big).$$

Thus,

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\big(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\beta\delta} g_{\alpha\gamma} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\alpha\gamma} g_{\beta\delta}\big) + g\cdot\partial g\cdot\partial g.$$

Using the property (5) of the normal coordinate system, we can readily verify that

$$R_{\alpha\beta\gamma\delta}|_p = \frac{1}{2}\big(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\beta\delta} g_{\alpha\gamma} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\alpha\gamma} g_{\beta\delta}\big)|_p$$

and

$$(\nabla_\lambda R)_{\alpha\beta\gamma\delta}|_p = \frac{1}{2}\big(\partial^3_{\lambda\beta\gamma} g_{\alpha\delta} - \partial^3_{\lambda\beta\delta} g_{\alpha\gamma} + \partial^3_{\lambda\beta\gamma} - \partial^3_{\lambda\alpha\gamma} g_{\beta\delta}\big)|_p$$

The identities 1–2 follow directly from the above expression, while the Bianchi identities (i.e. identities 3–4) follow using (6).

Taking two contractions of the 2^{nd} Bianchi identity, and using the symmetries of R (and the fact that $\nabla g = 0$, $\mathrm{Ric}_{\mu\nu} = g^{\kappa\lambda} R_{\mu\kappa\nu\lambda} = -g^{\kappa\lambda} R_{\mu\kappa\lambda\nu}$ and $R = g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta}$), we can also calculate:

$$\begin{aligned} 0 &= g^{\alpha\epsilon} g^{\gamma\delta} \nabla_\alpha R_{\beta\gamma\delta\epsilon} + g^{\alpha\epsilon} g^{\gamma\delta} \nabla_\gamma R_{\alpha\beta\delta\epsilon} + g^{\alpha\epsilon} g^{\gamma\delta} \nabla_\beta R_{\gamma\alpha\delta\epsilon} \\ &= -g^{\alpha\epsilon}\nabla_\alpha \mathrm{Ric}_{\beta\epsilon} - g^{\gamma\delta}\nabla_\gamma \mathrm{Ric}_{\beta\delta} + \nabla_\beta R \\ &= -2g^{\mu\nu}\nabla_\mu \mathrm{Ric}_{\nu\beta} + \nabla_\beta R, \end{aligned}$$

which is exactly the identity that we wanted to prove.

(b) When $dim\mathcal{M} = 3$, it is easy to check using the symmetries of the Riemann curvature tensor R that all components $R_{\alpha\beta\gamma\delta}$ can be obtained from the following ones:

$$R_{1212},\ R_{1313},\ R_{1213},\ R_{1232},\ R_{1323},\ R_{2323}$$

(one way to select six such components is to note that the symmetries of R imply that any component with three repeated indices vanishes; then, we can first compute all components where the index 1 appears twice, then once and finally a component where 1 doesn't appear as an index). Moreover, it is also easy to check that no one of the above components can be obtained from any of the other five via permutations of the indices (and, hence, from the symmetries of $R_{\alpha\beta\gamma\delta}$). This is the same number of independent components as for the Ricci tensor $Ric_{\alpha\beta} = g^{\gamma\delta}R_{\alpha\gamma\beta\delta}$ in this dimension (since $Ric_{\alpha\beta}$ is essentially a symmetric 3×3 matrix, it has at most 6 independent components; it can be also checked that each of $\{Ric_{11}, Ric_{12}, Ric_{13}, Ric_{22}, Ric_{23}, Ric_{33}\}$ cannot be obtained from the other five using the symmetries of R). In fact, it is easy to check that we have the following 6×6 linear system of relations between the components of Ric and R (using, again, the symmetries of R to carry out the appropriate permutations of the indices):

$$\begin{aligned}
Ric_{11} &= g^{22}R_{1212} + 2g^{23}R_{1213} + g^{33}R_{1313},\\
Ric_{12} &= -g^{12}R_{1212} - g^{13}R_{1213} - g^{23}R_{1232} + g^{33}R_{1323},\\
Ric_{13} &= -g^{13}R_{1313} + g^{12}R_{1213} - g^{23}R_{1323} + g^{22}R_{1232}\\
Ric_{22} &= g^{11}R_{1212} + 2g^{13}R_{1232} + g^{33}R_{2323}\\
Ric_{23} &= -g^{23}R_{2323} + g^{22}R_{1232} + g^{13}R_{1323} + g^{11}R_{1213}\\
Ric_{33} &= g^{11}R_{1313} + 2g^{12}R_{1323} + g^{22}R_{2323}.
\end{aligned}$$

It can be readily checked that, provided $[g_{ij}]$ is non-degenerate, the above system is also non-degenerate, namely it can be solved to express $\{R_{1212}, R_{1313}, R_{1213}, R_{1232}, R_{1323}, R_{2323}\}$ linearly in terms of $\{Ric_{11}, Ric_{12}, Ric_{13}, Ric_{22}, Ric_{23}, Ric_{33}\}$. In particular, if $Ric = 0$, then $R = 0$ as well.

In the case when $dim\mathcal{M} = 2$, it can be readily verified that R and Ric have each only one independent component (namely R_{1212} and R_{11}, since any component of R with three repeated indices has to vanish).

(c) Using the fact that $Ric_{\alpha\beta} = Ric_{\beta\alpha}$ and

$$\begin{aligned}
&R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma},\\
&R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta},\\
&R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0,
\end{aligned}$$

we can readily check that the Weyl tensor W also satisfies

$$\begin{aligned}
&W_{\alpha\beta\gamma\delta} = -W_{\alpha\beta\delta\gamma},\\
&W_{\alpha\beta\gamma\delta} = W_{\gamma\delta\alpha\beta},\\
&W_{\alpha\beta\gamma\delta} + W_{\alpha\delta\beta\gamma} + W_{\alpha\gamma\delta\beta} = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
g^{\gamma\delta}W_{\alpha\gamma\beta\delta} = g^{\gamma\delta}R_{\alpha\gamma\beta\delta} &+ \frac{1}{n-2}g^{\gamma\delta}\Big(Ric_{\alpha\delta}g_{\beta\gamma} - Ric_{\alpha\beta}g_{\gamma\delta} + Ric_{\beta\gamma}g_{\alpha\delta} - Ric_{\gamma\delta}g_{\alpha\beta}\Big)\\
&+ \frac{1}{(n-1)(n-2)}Rg^{\gamma\delta}\big(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma}\big)\\
= Ric_{\alpha\beta} &+ \frac{1}{n-2}\Big(Ric_{\alpha\delta}\delta^{\delta}_{\beta} - nRic_{\alpha\beta} + Ric_{\beta\gamma}\delta^{\gamma}_{\alpha} - Rg_{\alpha\beta}\Big)\\
&+ \frac{1}{(n-1)(n-2)}R\big(ng_{\alpha\beta} - \delta^{\gamma}_{\alpha}g_{\beta\gamma}\big)
\end{aligned}$$

$$
\begin{aligned}
&= Ric_{\alpha\beta} + \frac{1}{n-2}\Big(Ric_{\alpha\beta} - nRic_{\alpha\beta} + Ric_{\alpha\beta} - Rg_{\alpha\beta}\Big) \\
&\qquad + \frac{1}{(n-1)(n-2)}R\big((n-1)g_{\alpha\beta}\big) \\
&= 0.
\end{aligned}
$$

In the case when $dim\mathcal{M} = 3$, the fact that $W_{\alpha\beta\gamma\delta}$ satisfies the same symmetries as $R_{\alpha\beta\gamma\delta}$ implies that, exactly as in the case of part (a) of this exercise, all independent components of $W_{\alpha\beta\gamma\delta}$ can be expressed linearly in terms of the independent components of $g^{\gamma\delta}W_{\alpha\gamma\beta\delta}$; since the latter tensor vanishes identically in any dimension, we infer that $W = 0$ when $n = 3$.

(d) In any local coordinate chart on $\mathcal{M}$, we have

$$\tilde{g}_{\alpha\beta} = \phi^2 g_{\alpha\beta} \quad \text{and} \quad \tilde{g}^{\alpha\beta} = \phi^{-2}g^{\alpha\beta}.$$

We can, therefore, readily compute that the Christoffel symbols $\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2}\tilde{g}^{\gamma\delta}\big(\partial_\alpha\tilde{g}_{\delta\beta} + \partial_\beta\tilde{g}_{\delta\alpha} - \partial_\delta\tilde{g}_{\alpha\beta}\big)$ and $\Gamma^{\gamma}_{\alpha\beta}$ of $\tilde{g}$ and g, respectively, are related by:

$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + \frac{1}{\phi}\partial_\alpha\phi\delta^{\gamma}_{\beta} + \frac{1}{\phi}\partial_\beta\phi\delta^{\gamma}_{\alpha} - \frac{1}{\phi}\partial_\delta\phi g^{\gamma\delta}g_{\alpha\beta}.$$

Using the formula

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^{\alpha}_{\delta\beta} - \partial_\delta\Gamma^{\alpha}_{\gamma\beta} + \Gamma^{\alpha}_{\gamma\lambda}\Gamma^{\lambda}_{\delta\beta} - \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\gamma\beta}$$

for the components of the Riemann curvature tensor, we can similarly compute that

$$\tilde{R}_{\alpha\beta\gamma\delta} = \phi^2 R_{\alpha\beta\gamma\delta} - \phi^2\big(g_{\alpha\gamma}T_{\beta\delta} - g_{\beta\gamma}T_{\alpha\delta} + g_{\beta\delta}T_{\alpha\gamma} - g_{\alpha\delta}T_{\delta\gamma}\big), \tag{7}$$

where the tensor T is defined in terms of the conformal factor ϕ by

$$T_{\alpha\beta} = \phi^{-1}\partial^2_{\alpha\beta}\phi - \phi^{-1}\Gamma^{\lambda}_{\alpha\beta}\partial_\lambda\phi - 2\phi^{-2}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}\phi^{-2}\partial_\gamma\phi\partial_\delta\phi g^{\gamma\delta}g_{\alpha\beta}.$$

We can therefore readily check that the term $\bar{T}_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}T_{\beta\delta} - g_{\beta\gamma}T_{\alpha\delta} + g_{\beta\delta}T_{\alpha\gamma} - g_{\alpha\delta}T_{\delta\gamma}$ in the right hand side of (7) is traceless with respect to any pair of indices (namely any contraction of the form $g^{\alpha\gamma}\bar{T}_{\alpha\beta\gamma\delta}$ vanishes). Therefore, using the formula defining $W_{\alpha\beta\gamma\delta}$ in terms of $R_{\alpha\beta\gamma\delta}$ that, when considering the difference between $\tilde{W}_{\alpha\beta\gamma\delta}$ and $\phi^2 W_{\alpha\beta\gamma\delta}$ all the terms involving the tensor T from (7) cancel out, i.e.

$$\tilde{W}_{\alpha\beta\gamma\delta} - \phi^2 W_{\alpha\beta\gamma\delta} = 0.$$

In particular, since $R = 0$ (and, thus, $W = 0$) for the flat metric η, we infer that a metric g which is of the form $\phi^2 \cdot \eta$ has vanishing Weyl tensor.